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Electromagnetic Scalar Potentials at a Dielectric Interface

by Martin J. Lahart

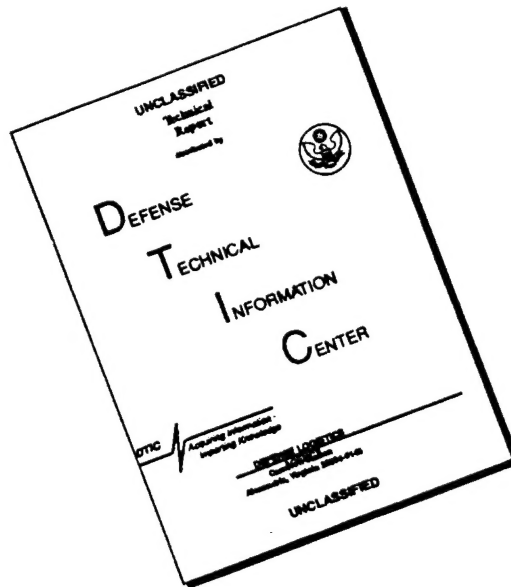ARL-TR-1177   October 1996

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13. ABSTRACT *(Maximum 200 words)*

  The use of scalar potentials in the solution of electromagnetic boundary value problems is described. Explicit solutions of fields at a dielectric interface and at a perfect conductor are derived in terms of the solutions to the wave equation that obey Dirichlet and Neumann boundary conditions. Application of a finiteness condition and a radiation condition are discussed. As an example of the approach, the Fresnel equations are derived. The scaler potentials used here are compared with the Debye potentials and the coupled azimuthal potentials, which have been used for similar applications.

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1. Introduction

The computation of electric and magnetic fields at dielectric boundaries must take into account interrelated boundary conditions, two involving normal components of the electric and magnetic fields and four involving tangential components of these fields. In a few simple cases, it is possible to determine the configuration of the polarization or propagation modes from general principles and to apply boundary conditions separately to each mode. For example, in the derivation of the Fresnel equations, which describe refraction of a plane wave at a planar surface, boundary conditions are applied separately to the s and p polarizations, with the electric or the magnetic fields in the plane of the surface. There are only three boundary conditions for each polarization, and the magnitudes and propagation directions for each are determined readily. Similarly, the lowest-order propagation modes of a dielectric fiber, which have no azimuthal field variations, are transverse electric and transverse magnetic [1], each with only three components; the magnitudes of the fields and propagation constants can be computed separately for each mode.

When the modal configurations are not known, two approaches can be taken to determine the fields. In one approach, four independent field configurations that account for all possibilities permitted by Maxwell's equations are constructed in such a way that the corresponding field components—tangential and normal electric and magnetic components—on each side of the interface are proportional everywhere on the interface. The continuity conditions at the boundary are applied to four of the components (e.g., the two tangential components of the electric and magnetic fields) to determine relative magnitudes of the four field configurations. This approach was used by Mie [2] to describe scattering from a dielectric sphere, and by Hondros [3] and Hondros and Debye [4] to describe propagation in a dielectric fiber.

The other approach uses electromagnetic scalar potentials, which are pairs of scalar solutions to the wave equation from which the electric and magnetic fields can be derived. Debye [5] used a pair of potentials (known today as the Debye potentials) to describe scattering from a dielectric sphere, which is the same problem that was solved by Mie. Debye developed expressions for the fields in terms of these functions, and he described boundary conditions that the scalar functions and their first derivatives were to satisfy on the interface (see eq (68)). To satisfy these conditions, Debye found scalar functions in which the values of the two functions and their normal derivatives were proportional on each side of the interface.

In addition to the Debye potentials, other pairs of electromagnetic scalar potentials have been investigated. Whittaker [6], who appears to have been the first person to realize that electromagnetic fields could be derived from two scalar potentials, described a pair from which fields due to moving point charges could be computed. Green and Wolf [7] used a different pair of potentials to compute the energy and momentum of electromagnetic fields and to construct a Lagrangian. Nisbet [8] developed a general transformation that relates possible pairs of scalar potentials, and Stratton [9] gave a simple expression for magnetic vector potentials in terms of an arbitrary set of scalar potentials (see eq (66)).

More recently, a representation of electromagnetic fields in terms of a different pair of scalar functions was developed by Morgan, et al [10] and extended by Morgan [11] for computing propagation through nonuniform dielectrics. These scalar functions—coupled azimuthal potentials—are not electromagnetic scalar potentials in the sense discussed above, because they do not independently satisfy the wave equation. Instead, they are solutions to a pair of coupled, second-order differential equations. Solutions are found by assuming Dirichlet boundary conditions at a geometric boundary of a body and computed across a dielectric interface somewhere within the body by a finite-difference method that begins at the geometric boundary. Moser, et al [12] used coupled azimuthal potentials to calculate frequencies of scattering resonances of bodies of revolution for application to target recognition. Several other applications are referenced in the paper by Morgan [11].

The requirement that both the scalar functions and their derivatives be proportional everywhere on each side of a dielectric interface, such as the arrangement represented for the Debye potentials in equation (68), is difficult to satisfy for complex geometries. Because of the spherical symmetry of his scattering object, Debye was able to express his scalar potentials in terms of spherical wave functions, and to satisfy the four continuity conditions with those functions. Most often, however, the forms of the functions are not known from general considerations, and a search for functions that match at the interface can be difficult.

In this report, it is shown that the scalar function that is zero at the surface and the one whose normal derivative is zero can be used as scalar potentials. Using these functions reduces the number of proportionality conditions at the boundary that must be considered from four to two, which relates the normal derivative of the first function and the value of the second function on each side of the interface. Boundary conditions at the interface are expressed in terms of these functions, and expressions for the electric and magnetic fields in terms of these functions are so derived.

It should be pointed out that the scalar potentials that satisfy the Dirichlet and Neumann conditions are used implicitly in the solution of boundary value problems involving perfect conductors. Here, a tangential component of the electric field is equated with a scalar function that is zero on the conducting surface, and a similar tangential component of the magnetic field is equated with a scalar function whose normal derivative is zero. For example, in the determination of fields in a conducting waveguide, the longitudinal electric and magnetic fields can be represented by the two scalar functions, and the transverse fields can be derived from these functions.

In the following sections, a set of equations is derived that describes boundary conditions in terms of magnetic vector potentials. The vector potentials are expressed in terms of scalar functions that obey the Dirichlet and Neumann boundary conditions. Solutions in terms of the scalar functions are found for the equations that describe boundary conditions at a dielectric interface. These solutions are combined in a way that satisfies the

radiation condition. There are two possible combinations, which are identified with propagation modes or polarizations. Through the equations that relate electric and magnetic fields to the vector potentials, these fields are also expressed in terms of the scalar functions. To demonstrate this approach, the methodology is used to derive the Fresnel equations. The methodology is also applied to find fields bounded by a conducting surface where, unlike the dielectric interface, there is no continuity requirement at the boundary, and differences between the two situations are discussed. Finally, the scalar potentials defined in this paper are compared with those used by Debye [5] and by Morgan, et al [10], and similarities and differences are discussed.

A two-dimensional approach is taken here by representing fields and potentials as products of functions of two coordinates and the exponential $\exp(-i\omega_z z)$, where z is the third coordinate. The results are directly applicable to a uniform waveguide (where the third coordinate is the longitudinal distance), or to a rotationally symmetric scatterer (where the third coordinate is in the azimuthal direction about the axis of symmetry).

2. Boundary Conditions for Vector Potentials

The boundary conditions for the vector potentials arise in a natural way by multiplying them by step functions that change from zero to one at the interface, and taking derivatives in the usual way. The coefficients of the delta function that result from the differentiation are the boundary conditions, and the coefficients of the step function make up the homogeneous wave equation for regions of constant permittivity and permeability. This approach was demonstrated for the electric and magnetic fields by von Roos [13] and is applied here, in a similar way, to sets of vector potentials that terminate at a conducting surface, and which are transmitted through a dielectric interface.

Consider first the fields produced on a perfectly conducting surface. Assume initially that the surface is planar, with a coordinate system shown in figure 1. The normal to the surface is in the x-direction. In terms of vector potentials, the Maxwell's equations for the fields are

$$\nabla \times \nabla \times A + \varepsilon_o \mu \frac{\partial^2 A}{\partial t^2} + \varepsilon_o \mu \frac{\partial}{\partial t} \nabla \Phi = \mu j \, \delta(x) \ , \tag{1}$$

and

$$\nabla^2 \Phi + \nabla \cdot \frac{\partial A}{\partial t} = -\frac{1}{\varepsilon_o} \rho \delta(x) \ , \tag{2}$$

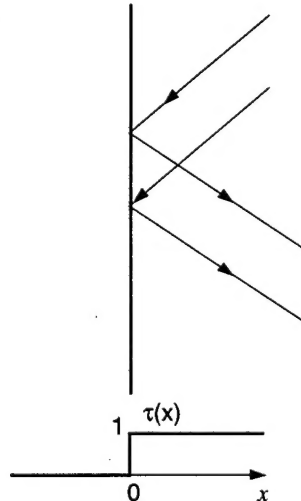where $\delta(x)$ is a delta function, and (A, Φ) are products of these solutions and the step function:

$$A(x, y, z) = \tau(x) A_o(x, y, z) \ ,$$
$$\Phi(x, y, z) = \tau(x) \Phi_o(x, y, z) \ , \tag{3}$$

where (A_o, Φ_o) are solutions to the homogenous equations that result when the right sides of equations (1) and (2) are zero.

The step function is defined as

$$\begin{aligned} \tau(x) &= 0 && \text{if } x < 0 \ , \\ &= 1 && \text{if } x > 0 \ . \end{aligned} \tag{4}$$

Figure 1. Directions of radiation incident on a conducting surface.

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Substituting equation (3) into equation (1) and applying vector identities gives

$$\left\{ \nabla^2 A_o - \varepsilon_o \mu \frac{\partial^2 A_o}{\partial t^2} - \nabla \left[\nabla \cdot A_o + \varepsilon_o \mu \frac{\partial \Phi_o}{\partial t} \right] \right\} \tau(x)$$
$$+ \left[A_{yo} \frac{\partial^2 \tau}{\partial x^2} + 2 \frac{\partial A_{yo}}{\partial x} \frac{d\tau}{dx} - \frac{\partial A_{xo}}{\partial y} \frac{d\tau}{dx} \right] \hat{y}$$
$$+ \left[A_{zo} \frac{\partial^2 \tau}{\partial x^2} + 2 \frac{\partial A_{zo}}{\partial x} \frac{d\tau}{dx} - \frac{\partial A_{xo}}{\partial z} \frac{d\tau}{dx} \right] \hat{z} = -\mu j \delta(x) \ , \tag{5}$$

where A_{yo} is the y-component of A_o and, similarly, for the other coordinates. In like manner, equation (2) may be expressed as

$$\left\{ \nabla^2 \Phi_o - \varepsilon_o \mu \frac{\partial^2 \Phi_o}{\partial t^2} + \frac{\partial}{\partial t} \left[\nabla \cdot A_o + \varepsilon_o \mu \frac{\partial \Phi_o}{\partial t} \right] \right\} \tau(x)$$
$$+ \left[\Phi_o \frac{d^2 \tau}{dx^2} + 2 \frac{\partial \Phi_o}{\partial x} \frac{d\tau}{dx} + \frac{\partial A_{xo}}{\partial t} \frac{d\tau}{dx} \right] = -\frac{1}{\varepsilon_o} \rho \delta(x) \ . \tag{6}$$

The first lines of equations (5) and (6) are identically zero, because the components of (A_o, Φ_o) satisfy the wave equation and, together, they obey the Lorentz condition

$$\frac{\partial A_{xo}}{\partial x} + \frac{\partial A_{yo}}{\partial y} + \frac{\partial A_{zo}}{\partial z} + \frac{1}{c^2} \frac{\partial \Phi_o}{\partial t} = 0 \ . \tag{7}$$

The other terms of equations (5) and (6) define the boundary conditions. To evaluate them, use the derivatives of τ, given by [14] as

$$\frac{d\tau}{dx} = \delta(x) \ ,$$
$$\frac{d^2 \tau}{dx^2} = -\frac{\delta(x)}{x} \ . \tag{8}$$

These terms become

$$\delta(x) \left[\frac{A_{yo}}{x} - 2 \frac{\partial A_{yo}}{\partial x} + \frac{\partial A_{xo}}{\partial y} \right] \hat{y} + \delta(x) \left[\frac{A_{zo}}{x} - 2 \frac{\partial A_{zo}}{\partial x} + \frac{\partial A_{xo}}{\partial z} \right] \hat{z} =$$
$$\delta(x) \mu \left(j_y \hat{y} + j_z \hat{z} \right) \tag{9}$$

and

$$\delta(x) \left[\frac{\Phi_o}{x} - 2 \frac{\partial \Phi_o}{\partial x} - \frac{\partial A_{xo}}{\partial t} \right] = \delta(x) \frac{\rho}{\varepsilon_o} \ . \tag{10}$$

Terms with x in the denominator have meaning only if the numerators are zero at the boundary. In this case, the ratio can be expressed by a derivative. With these substitutions, the boundary conditions become

$$\Phi_o = 0 \ ,$$
$$A_{zo} = 0 \ ,$$
$$A_{yo} = 0 \ ,$$

(11)

and

$$\frac{\partial \Phi_o}{\partial x} + \frac{\partial A_{xo}}{\partial t} = -\frac{1}{\varepsilon_o} \rho \ ,$$

$$\frac{\partial A_{zo}}{\partial x} - \frac{\partial A_{xo}}{\partial z} = -\mu j_z \ ,$$

$$\frac{\partial A_{yo}}{\partial x} - \frac{\partial A_{xo}}{\partial y} = -\mu j_y \ .$$

(12)

The corresponding boundary conditions at a dielectric interface are derived by setting the right sides of equations (1) and (2) to zero and representing the potentials and permittivity by

$$\Phi = (1 - \tau) \Phi_1 + \tau \Phi_2 \ ,$$
$$A = (1 - \tau) A_1 + \tau A_2 \ ,$$
$$\varepsilon = (1 - \tau) \varepsilon_1 + \tau \varepsilon_2 \ ,$$

(13)

where (A_1, Φ_1) and (A_2, Φ_2) are the solutions to the homogeneous wave equations in media 1 and 2, and ε_1 and ε_2 are the permittivities in the two media. Similar to the potentials described above, (A_1, Φ_1) and (A_2, Φ_2) obey the Lorentz condition. Applying vector identities as before and gathering coefficients of the delta functions gives the boundary conditions at the dielectric interface in terms of vector potentials

$$\Phi_1 = \Phi_2 \ ,$$
$$A_{z1} = A_{z2} \ ,$$
$$A_{y1} = A_{y2} \ ,$$

(14)

and

$$\varepsilon_1 \left[\frac{\partial \Phi_1}{\partial x} + \frac{\partial A_{x1}}{\partial t} \right] = \varepsilon_2 \left[\frac{\partial \Phi_2}{\partial x} + \frac{\partial A_{x2}}{\partial t} \right] \ ,$$

$$\left[\frac{\partial A_{z1}}{\partial x} - \frac{\partial A_{x1}}{\partial z} \right] = \left[\frac{\partial A_{z2}}{\partial x} - \frac{\partial A_{x2}}{\partial z} \right] \ ,$$

$$\left[\frac{\partial A_{y1}}{\partial x} - \frac{\partial A_{x1}}{\partial y} \right] = \left[\frac{\partial A_{y2}}{\partial x} - \frac{\partial A_{x2}}{\partial y} \right] \ .$$

(15)

These boundary conditions for the vector potentials are equivalent to the usual boundary conditions for electric and magnetic fields. Equation (14) is related to the continuity of the tangential components of the electric field E and the normal component of the magnetic induction B across the interface. Members of equation (15) are statements of the continuity of the normal component of the electric displacement D and the tangential components of the magnetic intensity H at the interface.

3. Scalar Functions at a Boundary Surface

Two sets of potentials will be computed separately. In the first set, three of the potentials are zero at the boundary and the normal derivative of the vector potential component normal to the surface is zero. These are the conditions of equation (11), and the potentials in this group are also the expressions for the potentials at a perfectly conducting interface. The second group of solutions obeys a set of conjugate boundary conditions for which the normal derivatives of the first three potentials and the fourth potential itself is zero. We express each of the vector potentials in terms of the functions

$$u(x, y) \exp\left[-i\left(\omega_z z - \omega t\right)\right] ,$$
$$v(x, y) \exp\left[-i\left(\omega_z z - \omega t\right)\right] , \tag{16}$$

where $u(x,y)$ and the normal derivative of $v(x,y)$ are zero on the boundary. Anticipating that we will want expressions for fields that are valid for surfaces of any orientation of the boundary in the (x,y) plane, we seek representations for vector potentials that are rotationally invariant in this plane. A set of expressions for the first group of solutions that satisfies these conditions is:

$$\Phi_1 = \phi_{11} u_1 ,$$
$$A_{z1} = a_{z11} u_1 ,$$
$$A_{y1} = id_{11} \frac{\partial u_1}{\partial y} + ia_{11} \frac{\partial v_1}{\partial x} ,$$
$$A_{x1} = id_{11} \frac{\partial u_1}{\partial x} - ia_{11} \frac{\partial v_1}{\partial y} , \tag{17}$$

where $(\phi_{11}, a_{z11}, d_{11}, a_{11})$ are constants. In these equations, the subscripts on the potentials (Φ_1, A_1) and on the scalar functions (u_1, v_1) refer to region 1. The constants have two numerical subscripts: the first subscript refers to the region, and the second subscript to the first or second set of solutions.

The coefficients $(\phi_{11}, a_{z11}, d_{11}, a_{11})$ are evaluated by substituting equation (17) in which they are regarded as variables into equation (15) to produce sets of linear equations. The Lorentz condition is used to eliminate d_{11}. In matrix form, the equations for the remaining coefficients are

$$\frac{\partial u_1}{\partial x} \begin{vmatrix} -\dfrac{\kappa_1 \omega_z^2}{k_1^2} & \dfrac{\kappa_1 \omega_z \omega}{k_1^2} & \omega \kappa_1 q_1 \\[2ex] -\dfrac{\kappa_1 \omega_z \omega/c^2}{k_1^2} & \dfrac{\kappa_1 \omega^2/c^2}{k_1^2} & \omega_z q_1 \\[2ex] 0 & 0 & \dfrac{k_1^2 v_1}{\partial u_1/\partial x} \end{vmatrix} \begin{vmatrix} \phi_{11} \\[2ex] a_{z11} \\[2ex] a_{11} \end{vmatrix} = \begin{vmatrix} V_1 \\[2ex] V_2 \\[2ex] V_3 \end{vmatrix} \tag{18}$$

for region 1 and, similarly, for region 2. Here κ_i and k_i^2 are the specific permittivity and the square of the magnitude of the propagation vector, given by

$$\kappa_i = \varepsilon_i / \varepsilon_0 \ ,$$

$$k_i^2 = \kappa_i \frac{\omega^2}{c^2} - \omega_z^2 \ , \tag{19}$$

and $q_1 = \dfrac{\partial v_1 / \partial y}{\partial u_1 / \partial x}$. We assume q_i is constant on both sides of the boundary. Without loss of generality, we can normalize (u,v) so that their derivatives are equal at the boundary

$$\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} \ ,$$

$$\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial y} \ . \tag{20}$$

As a consequence of this normalization, $v_1 = v_2$ and $q_1 = q_2$ hold.

The only quantities in equation (18) that are characteristic of the medium are κ_i and k_i^2. Equation (18) can be satisfied by solutions that lead to (V_1, V_2, V_3) that are not functions of either of these quantities. A set of solutions for which this is true is

$$\begin{bmatrix} \phi_{i1} \\ a_{zi1} \\ a_{i1} \end{bmatrix} = \begin{bmatrix} \omega/c \\ \omega_z/c \\ 0 \end{bmatrix} , \begin{bmatrix} (1/\kappa_i)\omega_z \\ \omega/c^2 \\ 0 \end{bmatrix} , \begin{bmatrix} (1/\kappa_i)\omega_z q \\ 0 \\ (\omega\omega_z)/(ck_i)^2 \end{bmatrix} , \tag{21}$$

where i is 1 or 2. The first of these solutions defines gauge transformations and the other two lead to nonzero fields.

The second group of solutions to equation (15) can be written in terms of (u,v) as

$$\Phi_1 = \phi_{12} v_1 \ ,$$

$$A_{z1} = a_{z12} v_1 \ ,$$

$$A_{y1} = i d_{12} \frac{\partial v_1}{\partial y} + i a_{12} \frac{\partial u_1}{\partial x} \ , \tag{22}$$

$$A_{x1} = i d_{12} \frac{\partial v_1}{\partial x} - i a_{12} \frac{\partial u_1}{\partial y}$$

for region 1 and, similarly, for region 2. As in equation (17), these solutions are rotationally invariant in the (x,y) plane. Their substitution into equation (15) leads to the following equations for the constants

$$\varepsilon_1\left[\phi_{12}\frac{\partial v_1}{\partial x}-\omega a_{12}\frac{\partial u_1}{\partial y}-\omega d_{12}\frac{\partial v_1}{\partial x}\right]=\varepsilon_2\left[\phi_{22}\frac{\partial v_2}{\partial x}-\omega a_{22}\frac{\partial u_2}{\partial y}-\omega d_{22}\frac{\partial v_2}{\partial x}\right],$$

$$\left[a_{z12}\frac{\partial v_1}{\partial x}-\omega_z a_{12}\frac{\partial u_1}{\partial y}-\omega_z d_{12}\frac{\partial v_1}{\partial x}\right]=\left[a_{z22}\frac{\partial v_2}{\partial x}-\omega_z a_{22}\frac{\partial u_2}{\partial y}-\omega_z d_{22}\frac{\partial v_2}{\partial x}\right],$$

$$a_{12}\frac{\partial^2 u_1}{\partial x^2}+d_{12}\frac{\partial^2 v_1}{\partial x\partial y}=a_{22}\frac{\partial^2 u_2}{\partial x^2}+d_{22}\frac{\partial^2 v_2}{\partial x\partial y}\ .$$

(23)

Both sides of each of these equations are identically zero because the derivatives of u and v that appear in these equations are zero at the boundary. The boundary conditions represented in equation (14), which were identically zero for the potentials of equation (17), are used to evaluate the constants in equation (22). Substitution of equation (22) into the first three members of equation (14) gives:

$$\phi_{12}v_1=\phi_{22}v_2\ ,$$

$$a_{z12}v_1=a_{z22}v_2\ ,$$

$$d_{12}\frac{\partial v_1}{\partial y}+a_{12}\frac{\partial u_1}{\partial x}=d_{22}\frac{\partial v_2}{\partial y}+a_{22}\frac{\partial u_2}{\partial x}\ .$$

(24)

The Lorentz condition may be used to eliminate one of the constants, as before. The resulting equations can be expressed in matrix form as

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \dfrac{\kappa_i\!\left(\omega/c^2\right)q}{k_i^2} & -\dfrac{\omega_z q}{k_i^2} & 1\end{bmatrix}\begin{bmatrix}\phi_{i2}\\ a_{zi2}\\ a_{i2}\end{bmatrix}=\begin{bmatrix}W_1\\ W_2\\ W_3\end{bmatrix}.$$

(25)

As with equation (18), equation (25) may be satisfied by sets of constants that yield values of (W_1,W_2,W_3) that are not functions of κ_i or k_i^2. A set of linearly independent solutions is

$$\begin{bmatrix}\phi_{i2}\\ a_{z12}\\ a_{12}\end{bmatrix}=\begin{bmatrix}\omega/c\\ \omega_z/c\\ 0\end{bmatrix},\begin{bmatrix}0\\ \omega/c^2\\ \dfrac{\left(\omega/c^2\right)}{\omega_z}\left(1+\dfrac{\omega_z^2 q}{k_i^2}\right)\end{bmatrix},\begin{bmatrix}0\\ 0\\ \dfrac{\omega/c^2}{\omega_z}\end{bmatrix}.$$

(26)

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As with equation (21), the first of these solutions defines gauge transformations and the others define propagation modes.

The electric and magnetic fields can be computed by substituting the potentials described by equations (17) and (22) into the equations for the electric and magnetic fields, which becomes

$$B = \nabla \times A ,$$

$$E = - \nabla \Phi - \frac{\partial A}{\partial t} . \tag{27}$$

It is convenient to define the quantities

$$b_i = -\left(a_{i1} v_i + a_{i2} u_i\right)$$

$$e_i = \left(1/k_i^2\right)\left[\left(\omega_z \phi_{i1} - \omega a_{zi1}\right) u_i + \left(\omega_z \phi_{i2} - \omega a_{zi2}\right) v_i\right] , \tag{28}$$

where the subscript i denotes regions 1 or 2. In terms of these quantities, the electric and magnetic fields are

$$B_{xi} = \omega_z \frac{\partial b_i}{\partial x} - \frac{\kappa_i \omega}{c^2} \frac{\partial e_i}{\partial y} , \qquad E_{xi} = \omega \frac{\partial b_i}{\partial y} + \omega_z \frac{\partial e_i}{\partial x} ,$$

$$B_{yi} = \omega_z \frac{\partial b_i}{\partial y} + \frac{\kappa_i \omega}{c^2} \frac{\partial e_i}{\partial x} , \qquad E_{yi} = - \omega \frac{\partial b_i}{\partial x} + \omega_z \frac{\partial e_i}{\partial y} ,$$

$$B_{zi} = ik_i^2 b_i , \qquad\qquad E_{zi} = ik_i^2 e_i : \tag{29}$$

Substitution of the first solutions of equations (21) and (26) into equation (29) yields values of b_i and e_i that are zero, which verifies that these solutions define gauge transformations.

General expressions for the sets of constants in equations (21) and (26) that lead to nonzero fields may be constructed as linear combinations of the second and third solutions:

$$\begin{bmatrix} \phi_{i1} \\ a_{iz1} \\ a_{i1} \end{bmatrix} = t_1 \begin{bmatrix} (1/\kappa_i)\, \omega_z \\ \omega/c^2 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} (1/\kappa_i)\, \omega_z q \\ 0 \\ (\omega\omega_z)/(ck_i)^2 \end{bmatrix} \tag{30}$$

and

$$\begin{bmatrix} \phi_{i2} \\ a_{iz2} \\ a_{i2} \end{bmatrix} = t_3 \begin{bmatrix} 0 \\ \omega/c \\ \dfrac{(\omega/c^2)}{\omega_z}\left(1 + \dfrac{\omega_z^2 q}{k_i^2}\right) \end{bmatrix} + t_4 \begin{bmatrix} 0 \\ 0 \\ \dfrac{\omega/c^2}{\omega_z} \end{bmatrix} , \tag{31}$$

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where (t_1, t_2, t_3, t_4) are constants. Substitution of equations (30) and (31) into equation (28) leads to expressions for b_i and e_i in terms of (t_1, t_2, t_3, t_4):

$$b_i = -\frac{\omega}{c^2}\frac{1}{\omega_z}\left[t_3\left(1 + \frac{\omega_z^2 q}{k_i^2}\right) + t_4\right]u_i - \frac{\omega}{c^2}\frac{\omega_z}{k_i^2}t_2 v_i \ ,$$

$$e_i = -\frac{1}{\kappa_i}\left[t_1 - t_2\frac{\omega_z^2 q}{k_i^2}\right]u_i - \frac{1}{k_i^2}\frac{\omega^2}{c^2}t_3 v_i \ . \tag{32}$$

Because the values of (b_i, e_i) are independent of the orientation of the coordinate system in which (u_i, v_i) were defined, equation (32) holds for any point on the interface, provided that q has the same value at every point; the functions u and v must be divided into components for which this is the case. For example, if the interface is planar, u and v might be

$$u \sim u(x)\exp\left(-i\,\omega_y y\right) \ ,$$

$$v \sim v(x)\exp\left(-i\,\omega_y y\right) \ . \tag{33}$$

The ratio q would be

$$q = \frac{-i\,\omega_y v(x)}{\partial u(x)/\partial x} \ . \tag{34}$$

The ratio q is a function of ω_y, implying that propagation across the interface would be different for components of different ω_y. The condition that q be constant and the conditions of equation (20) are requirements for the functions u and v at the interface.

It was assumed initially that the boundary of the region where fields are calculated was planar. Since the fields in this two-dimensional problem are proportional to u and v or their first derivatives, this restriction can be dropped because surface curvature enters only in the computation of second derivatives.

4. Radiation Condition

In addition to the boundary conditions at the dielectric interface, another condition must be satisfied: radiation that propagates across the interface must be outgoing or exponentially decaying. In the configuration of figure 2, radiation is incident from region 1 and radiation outgoing in region 2. This configuration places constraints on possible values of (t_1, t_2, t_3, t_4). To calculate these constants, we can express u_2 and v_2 as functions of outgoing and incoming radiation functions f and g as

$$u_2 = \alpha_o f + \alpha g \ ,$$
$$v_2 + \beta_o f + \beta g \ , \tag{35}$$

where f represents outgoing or exponentially decaying radiation and g represents incoming or exponentially growing radiation. The quantities b_2 and e_2 become

$$b_2 = -\left\{ \alpha_o \frac{\omega}{c^2} \frac{1}{\omega_z} \left[t_3 \left(1 + \frac{\omega_z^2 q}{k_2^2} \right) + t_4 \right] + \beta_o \frac{\omega}{c^2} \frac{\omega_z}{k_2^2} t_2 \right\} f$$
$$- \left\{ \alpha \frac{\omega}{c^2} \frac{1}{\omega_z} \left[t_3 \left(1 + \frac{\omega_z}{k_2^2} \right) + t_4 \right] + \beta \frac{\omega}{c^2} \frac{\omega_z}{k_2^2} t_2 \right\} g \ , \tag{36}$$

and

$$e_2 = -\left\{ \alpha_o \frac{1}{\kappa_2} \left[t_1 - t_2 \frac{\omega_z^2 q}{k_2^2} \right] - \beta_o \frac{1}{k_2^2} \frac{\omega^2}{c^2} t_3 \right\} f$$
$$- \left\{ \alpha \frac{1}{\kappa_2} \left[t_1 - t_2 \frac{\omega_z^2 q}{k_2^2} \right] - \beta \frac{1}{k_2^2} \frac{\omega^2}{c^2} t_3 \right\} g \ . \tag{37}$$

Figure 2. Local coordinate system on interface between two dielectrics.

region 2
$\varepsilon = \varepsilon_2$

region 1
$\varepsilon = \varepsilon_1$

y

x
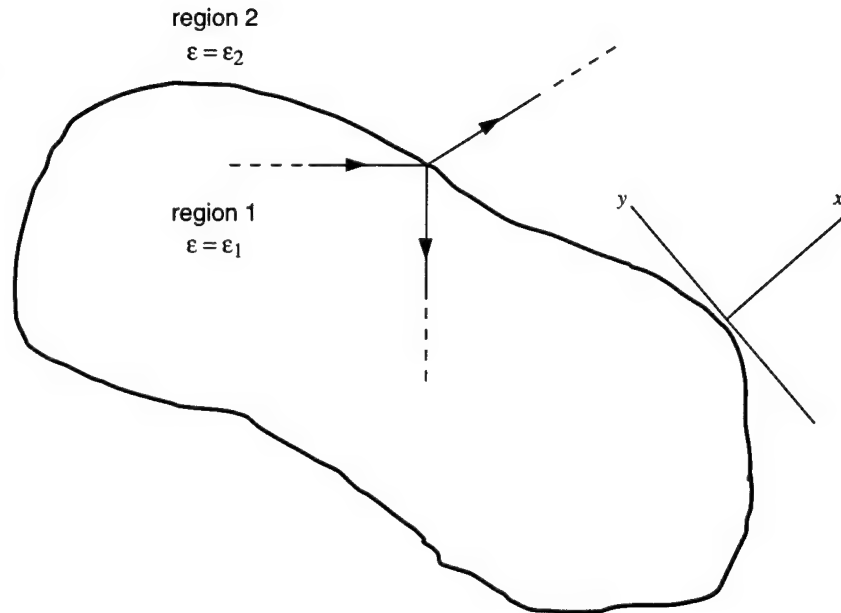
The coefficients of g must be zero. Equations for α and β are

$$\alpha \frac{\omega}{c^2} \frac{1}{\omega_z} \left[t_3 \left(1 + \frac{\omega_z}{k_2^2} \right) + t_4 \right] + \beta \frac{\omega}{c^2} \frac{\omega_z}{k_2^2} t_2 = 0 \ ,$$

$$\alpha \frac{1}{\kappa_2} \left[t_1 - t_2 \frac{\omega_z^2 q}{k_2^2} \right] + \beta \frac{1}{k_2^2} \frac{\omega^2}{c^2} t_3 = 0 \ . \tag{38}$$

Because these equations are homogeneous, they can only have nonzero solutions for α and β if the coefficients of α and β in each equation are proportional. Let T be the constant of proportionality, defined by

$$\omega_z^2 t_2 = T \frac{\kappa_2 \omega^2}{c^2} t_3 \ . \tag{39}$$

In terms of T, equation (38) is

$$\alpha \left[t_3 \left(1 + \frac{\omega_z}{k_2^2} \right) + t_4 \right] + \beta T \frac{\kappa_2 \left(\omega^2/c^2 \right)}{k_2^2} t_3 = 0 \ ,$$

$$\alpha \left[t_1 - \frac{\kappa_2 \left(\omega^2/c^2 \right) q}{k_2^2} T t_3 \right] + \beta \frac{\kappa_2 \left(\omega^2/c^2 \right)}{k_2^2} t_3 = 0 \ . \tag{40}$$

The coefficients of α must also be proportional, as in

$$t_3 \left(1 + \frac{\omega_z^2}{k_2^2} \right) + t_4 = t_1 - \frac{\kappa_2 \left(\omega^2/c^2 \right) q}{k_2^2} t_3 T \ . \tag{41}$$

Equations (39) and (41) may be used to eliminate t_2 and t_4 from the expressions for b_2 and e_2. Equation (40) may be used to express t_1 in terms of the ratio β/α. These expressions for (t_1, t_2, t_3, t_4) may be substituted into expressions for b_1 and e_1, which are similar to equations (36) and (37), with k_2 replaced by k_1. The result is

$$b_1 = -\frac{\omega}{c^2} \frac{1}{\omega_z} \left[q \left(\frac{\kappa_1}{k_1^2} - \frac{\kappa_2}{k_2^2} \right) - T \frac{\kappa_2}{k_2^2} \frac{\beta}{\alpha} \right] u_1 - T \frac{\kappa_2 \left(\omega/c^2 \right)}{k_1^2 \omega_z} v_1 \ ,$$

$$e_1 = \frac{\kappa_2}{\kappa_1} \left[q T \left(\frac{1}{k_1^2} - \frac{1}{k_2^2} \right) + \frac{1}{k_2^2} \frac{\beta}{\alpha} \right] u_1 + \frac{1}{k_1^2} v_1 \ . \tag{42}$$

A similar expression for medium 2 is

$$b_2 = T \frac{\omega}{c^2} \frac{1}{\omega_z} \frac{\kappa_2}{k_2^2} \left(\frac{\beta}{\alpha} u_2 - v_2 \right) \ ,$$

$$e_2 = \frac{1}{k_2^2} \left(\frac{\beta}{\alpha} u_2 - v_2 \right) \ . \tag{43}$$

17

All the electric and magnetic fields may be derived from (b, e) using equation (29). The parameter T defines propagation modes. It is often convenient to define modes with $T = 0$ and $T = \infty$, but it is not necessary to do this. Any two values of T will define linearly independent sets of (b, e) in terms that may express all other possibilities.

Although this derivation was carried out with a radiation condition in mind, the result is more general. The procedure is valid because the two independent solutions (u, v) to the wave equation can be expressed as a linear combination of two different solutions (f, g). The functions (f, g) may represent solutions that are physically admissible or inadmissible. In this case, the admissible solutions were outgoing or exponentially decaying, and the inadmissible solutions were incoming or exponentially growing.

5. Fresnel Equations

The simplest interface between two dielectric media is a plane. Figure 3 shows incident and reflected waves in medium 1, and an outgoing refracted wave in medium 2. We assume that the waves are planar, with directions of propagation in the x-z plane. The radiation has no dependence on the y-coordinate. In medium 2, the functions u and v are

$$u_2 = a \times \exp\left(-i\omega_z z\right) \sin \omega_x x \ ,$$
$$v_2 = \exp\left(-i\omega_z z\right) \cos \omega_x x \ . \tag{44}$$

For outgoing and incoming radiation, the functions f and g, are

$$f = \exp\left[-i\left(\omega_z z + \omega_{x2} x\right)\right] \ ,$$
$$g = \exp\left[-i\left(\omega_z z - \omega_{x2} x\right)\right] \ . \tag{45}$$

The constants α and β are

$$\alpha = -i\left(a/2\right) \ ,$$
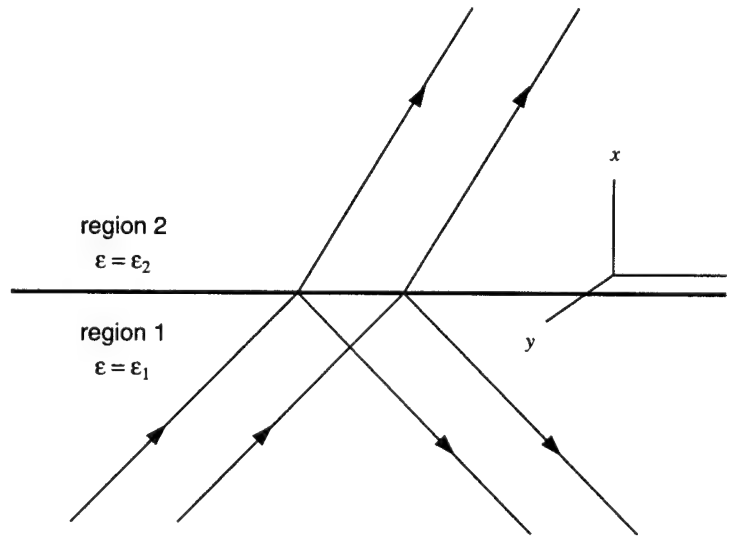$$\beta = 1/2 \ ,$$
$$\beta/\alpha = i/a \ . \tag{46}$$

Functions u_1 and v_1 in medium 1 are defined so that

$$\partial u_1/\partial x = \partial u_2/\partial x \ ,$$
$$v_1 = v_2 \ . \tag{47}$$

With these normalizations, u_1 and v_1 are

$$u_1 = a\left(\omega_{x2}/\omega_{x1}\right)\exp\left(-i\omega_z z\right)\sin \omega_{x1} x \ ,$$
$$v_1 = \exp\left(-i\omega_z z\right)\cos \omega_{x1} x \ . \tag{48}$$

Figure 3. Planar interface between two dielectrics and directions of propagation of radiation.

region 2
$\varepsilon = \varepsilon_2$

region 1
$\varepsilon = \varepsilon_1$

The ratio of derivatives q is zero, because v has no dependence on y. With this value of q, equations (42) and (43) become

$$b_1 = T\frac{\omega}{c^2}\frac{\kappa_2}{\omega_z}\left(\frac{i}{ak_2^2}u_1 - \frac{1}{k_1^2}v_1\right) ,$$

$$e_1 = \frac{\kappa_1}{\kappa_2}\left(\frac{i}{ak_2^2}u_1\right) - \frac{1}{k_1^2}v_1 , \qquad (49)$$

and

$$b_2 = T\frac{\omega}{c^2}\frac{\kappa_2}{\omega_z}\frac{1}{k_2^2}\left(\frac{i}{a}u_2 - v_2\right) ,$$

$$e_2 = \frac{1}{k_2^2}\left(\frac{i}{a}u_2 - v_2\right) . \qquad (50)$$

Substitution of equations (44) and (48) into these expressions gives

$$b_1 = T\frac{\omega}{c^2}\frac{\kappa_2}{\omega_z}\left(i\frac{\omega_{x2}}{\omega_{x1}}\frac{1}{k_2^2}\sin\omega_{x1}x - \frac{1}{k_1^2}\cos\omega_{x1}x\right)\exp\left(-i\omega_z z\right) ,$$

$$e_1 = \left(i\frac{\kappa_2}{\kappa_1}\frac{\omega_{x2}}{\omega_{x1}}\frac{1}{k_2^2}\sin\omega_{x1}x - \frac{1}{k_1^2}\cos\omega_{x1}x\right)\exp\left(-i\omega_z z\right) , \qquad (51)$$

and

$$b_2 = T\frac{\omega}{c^2}\frac{\kappa_2}{\omega_z}\frac{1}{k_2^2}\left(i\sin\omega_{x2}x - \cos\omega_{x2}x\right)\exp\left(-i\omega_z z\right) ,$$

$$e_2 = \frac{1}{k_2^2}\left(i\sin\omega_{x2}x - \cos\omega_{x2}x\right)\exp\left(-i\omega_z z\right) . \qquad (52)$$

The quantities (b_1, e_1) and (b_2, e_2) may be expressed in terms of incoming and outgoing waves as in

$$b_1 = -\frac{1}{2}T\frac{\kappa_2(\omega/c^2)}{\omega_z\omega_{x2}\omega_{x1}^2}\left[(\omega_{x2} - \omega_{x1})e^{i\omega_{x1}x} + (\omega_{x2} + \omega_{x1})e^{-i\omega_{x1}x}\right]\exp\left(-i\omega_z z\right) ,$$

$$e_1 = -\frac{1}{2\kappa_1\omega_{x2}\omega_{x1}^2}\left[(\kappa_1\omega_{x2} - \kappa_2\omega_{x1})e^{i\omega_{x1}x} + (\kappa_1\omega_{x2} + \kappa_2\omega_{x1})e^{-i\omega_{x1}x}\right]\exp\left(-i\omega_z z\right) , \qquad (53)$$

and

$$b_2 = -T\frac{\kappa_2(\omega/c^2)}{\omega_z\omega_{x2}^2}e^{-i\omega_{x2}x}\exp\left(-i\omega_z z\right) ,$$

$$e_2 = -\frac{1}{\omega_{x2}^2}e^{-i\omega_{x2}x}\exp\left(-i\omega_z z\right) . \qquad (54)$$

Because of the lack of dependence of (b, e) on y, the equations for the electric and magnetic fields given in equation (29) simplify to

$$B_x = \omega_z \frac{\partial b}{\partial x}, \qquad E_x = \omega_z \frac{\partial e}{\partial x},$$

$$B_y = \frac{\kappa \omega}{c^2} \frac{\partial e}{\partial x}, \qquad E_y = -\omega \frac{\partial b}{\partial x},$$

$$B_z = ik^2 b, \qquad E_z = ik^2 e,$$

(55)

where (κ, k^2) stand for (κ_1, k_1^2) or (κ_2, k_2^2), depending on the medium in which the fields are evaluated. Equations (53) and (54) are substituted into equation (55) to produce values of the B and E fields.

The parameter T defines the polarization modes. If T is large, e, by comparison, is small, and B_y, E_x, and E_z are, effectively, zero. This condition defines the mode in which the electric field is perpendicular to the plane of incidence and refraction. Apart from factors that are common to both regions, the fields for this mode in region 1 are

$$B_x = -\frac{1}{2} \omega_z \left(\omega_{x2} - \omega_{x1}\right) e^{i\omega_{x1} x} + \frac{1}{2} \omega_z \left(\omega_{x2} + \omega_{x1}\right) e^{-i\omega_{x1} x},$$

$$B_z = -\frac{1}{2} \omega_{x1} \left(\omega_{x2} - \omega_{x1}\right) e^{i\omega_{x1} x} - \frac{1}{2} \omega_{x1} \left(\omega_{x2} + \omega_{x1}\right) e^{-i\omega_{x1} x},$$

$$E_y = \frac{1}{2} \omega \left(\omega_{x2} - \omega_{x1}\right) e^{i\omega_{x1} x} - \frac{1}{2} \omega \left(\omega_{x2} + \omega_{x1}\right) e^{-i\omega_{x1} x},$$

(56)

and, in region 2, are

$$B_x = \omega_z \, \omega_{x2} \, e^{-i\omega_{x1} x},$$

$$B_z = -\omega_{x2}^2 \, e^{-i\omega_{x1} x},$$

$$E_y = -\omega \, \omega_{x2} \, e^{-i\omega_{x1} x}.$$

(57)

Likewise, if T is zero, b is zero in both media and B_x, B_z, and E_y are zero. This value of T defines the polarization mode in which the magnetic field is perpendicular to the plane of incidence and refraction. The fields for this mode in region 1 are

$$B_y = \frac{1}{2} \frac{\omega}{c} \left(\kappa_1 \, \omega_{x2} - \kappa_2 \, \omega_{x1}\right) e^{i\omega_{x1} x} - \frac{1}{2} \frac{\omega}{c} \left(\kappa_1 \, \omega_{x2} + \kappa_2 \, \omega_{x1}\right) e^{-i\omega_{x1} x},$$

$$E_x = \frac{1}{2} \frac{\omega_z c}{\kappa_1} \left(\kappa_1 \, \omega_{x2} - \kappa_2 \, \omega_{x1}\right) e^{i\omega_{x1} x} - \frac{1}{2} \frac{\omega_z c}{\kappa_1} \left(\kappa_1 \, \omega_{x2} + \kappa_2 \, \omega_{x1}\right) e^{-i\omega_{x1} x},$$

$$E_z = \frac{1}{2} \frac{\omega_{x1} c}{\kappa_1} \left(\kappa_1 \, \omega_{x2} - \kappa_2 \, \omega_{x1}\right) e^{i\omega_{x1} x} + \frac{\omega_{x1} c}{\kappa_1} \left(\kappa_1 \, \omega_{x2} + \kappa_2 \, \omega_{x1}\right) e^{-i\omega_{x1} x},$$

(58)

and, in region 2, are

$$B_y = -\kappa_2 \frac{\omega}{c} \omega_{x2} e^{-i\omega_{x1}x} ,$$

$$E_x = -\omega_z \omega_{x2} \alpha e^{-i\omega_{x1}x} , \tag{59}$$

$$E_z = \omega_{x2}^2 \alpha e^{-i\omega_{x1}x} .$$

The Fresnel equations are usually expressed as ratios of reflected and refracted fields to the incident fields, and the fields are expressed in terms of the angle between the direction of propagation and the surface normal. The geometry is illustrated in figure 4, which shows the wavefronts, the direction of propagation, and the coordinate system in which the x-coordinate is normal to the surface. The spatial frequencies ω_x and ω_z may be expressed in terms of the angle θ as

$$\omega_z = \sqrt{\kappa} \frac{\omega}{c} \sin \theta ,$$

$$\omega_x = \sqrt{\kappa} \frac{\omega}{c} \cos \theta . \tag{60}$$

Substitution of these relationships into equations (56) through (59) leads to the more common form of the Fresnel equations [15].

Figure 4. Definition of angle θ that direction of propagation makes with surface normal.
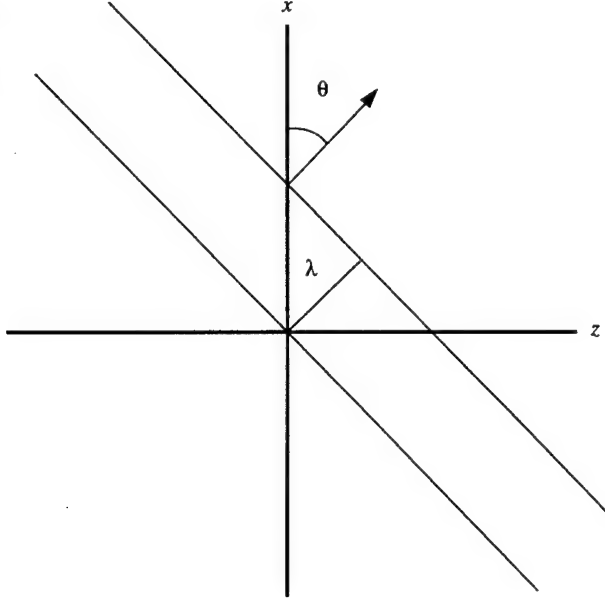
6. Boundary Conditions at a Conducting Surface

The first group of solutions for boundary conditions at a dielectric interface (eq (17)) can be used to describe boundary conditions at the surface of a conductor; these potentials satisfy the boundary conditions of equation (11). Substitution of equation (17) into equation (12) lead to linear equations for the constants

$$(\phi - \omega d)\frac{\partial u}{\partial x} - \omega a \frac{\partial v}{\partial y} = -\frac{\rho}{\varepsilon_0} \ ,$$

$$(a_z - \omega_z d)\frac{\partial u}{\partial x} - \omega_z a \frac{\partial v}{\partial y} = -\mu j_z \ , \qquad (61)$$

$$-ik^2 av = -\mu j_y \ ,$$

where the subscripts on (ϕ, a_z, d, a) that denote the region and the set of solutions have been dropped. The Lorentz condition is invoked, as before, to eliminate the constant d, leading to three equations for the three remaining constants. In matrix form, these equations are

$$\frac{\partial u}{\partial x}\begin{bmatrix} -\dfrac{\omega_z^2}{k^2} & \dfrac{\omega \omega_z}{k^2} & \omega q \\[2ex] -\dfrac{\omega_z(\omega/c^2)}{k^2} & \dfrac{\omega^2/c^2}{k^2} & \omega_z q \\[2ex] 0 & 0 & -ik^2\dfrac{v}{\partial u/\partial x} \end{bmatrix}\begin{bmatrix} \phi \\[2ex] a_z \\[2ex] a \end{bmatrix} = -\begin{bmatrix} \rho/\varepsilon_0 \\[2ex] \mu j_z \\[2ex] \mu j_y \end{bmatrix} \ , \qquad (62)$$

where, as before, $q = \dfrac{\partial v/\partial y}{\partial u/\partial x}$. A set of linearly independent solutions to the matrix equation is

$$\begin{bmatrix} \varphi \\ a_z \\ a \end{bmatrix} = \begin{bmatrix} \omega/c \\ \omega_z/c \\ 0 \end{bmatrix} \ , \quad \begin{bmatrix} \omega_z \\ \omega/c^2 \\ 0 \end{bmatrix} \ , \quad \begin{bmatrix} \omega_z q \\ \omega_z^2 q/\omega \\ -(\omega \omega_z)/(ck)^2 \end{bmatrix} \ . \qquad (63)$$

The first solution has an eigenvalue of zero and defines gauge transformations. The third solution differs from that of equation (21). It will be shown that these solutions define *TM* and *TE* propagation modes, where the direction of propagation is in the z-direction. The modes resulting from these solutions will be the same as those modes usually employed to analyze conducting waveguides; however, any pair of linearly independent solutions that lead to nonzero charges and currents, and that is also independent of the gauge solution, defines physical field configurations.

The electric and magnetic fields are computed as before by substituting equation (63) into equation (28). The fields are

$$B_x = -\frac{\omega}{c^2}\frac{\partial u}{\partial y} \; , \quad E_x = \omega_z \frac{\partial u}{\partial x} \; ,$$

$$B_y = \frac{\omega}{c^2}\frac{\partial u}{\partial x} \; , \quad E_y = \omega_z \frac{\partial u}{\partial y} \; ,$$

$$B_z = 0 \; , \quad\quad E_z = ik^2 u \; , \quad (64)$$

and

$$B_x = \frac{\omega}{c^2}\frac{\omega_z}{k^2}\frac{\partial v}{\partial x} \; , \quad E_x = \frac{\omega^2}{c^2}\frac{\omega_z}{k^2}\frac{\partial v}{\partial y} \; ,$$

$$B_y = \frac{\omega}{c^2}\frac{\omega_z}{k^2}\frac{\partial v}{\partial y} \; , \quad E_y = -\frac{\omega^2}{c^2}\frac{\omega_z}{k^2}\frac{\partial v}{\partial x} \; ,$$

$$B_z = i\frac{\omega\omega_z}{c^2}v \; , \quad\quad E_z = 0 \; . \quad (65)$$

Although the vector potentials were computed at a single point of the boundary, the only quantity that characterizes individual points (the ratio q) does not appear in the expressions for the fields. Unlike the dielectric interface, computations at any point on the conducting surface lead to the same equations for the fields, even if q is not constant. Moreover, as a consequence of the rotationally invariant forms in equation (17), the forms of equations (66) and (67) are also invariant to rotations in the (x, y) plane. These are the equations for the fields in a conducting waveguide [16], relative to any coordinate system whose z-axis is in the direction of propagation. With different variables, they also describe other two-dimensional problems.

7. Discussion

The three orthogonal magnetic vector potentials are represented as [9]

$$
\begin{aligned}
A_\xi &= \nabla \psi_i \\
A_\eta &= \nabla \times (\bar{s}\, \psi_i) \\
A_\zeta &= \nabla \times \nabla \times (\bar{s}\, \psi_i) \ ,
\end{aligned}
\tag{66}
$$

where $\bar{s}$ is an arbitrary constant vector that has been called the pilot vector [17] and ψ_i is a scalar solution to the wave equation. The electric potential Φ can be constructed from the other components of A_ξ, A_η, or A_ζ by using the Lorentz condition (eq (7)). The vector A_ξ and the potential Φ that is associated with it define gauge transformations; the electric and magnetic fields associated with this set of solutions are zero. For each Fourier component ω_z, the subscript in equation (66) can be 1 or 2, because there are two independent solutions to the scalar wave equation. Excluding the gauge solutions, four sets of solutions to Maxwell's equations are represented in equation (66). This representation is consistent with the four solutions in equations (30) and (31). Different sets of scalar potentials arise through choices of the pilot vector $\bar{s}$, the independent scalar solutions ψ_i, and the choice of gauge.

The Debye potentials can be related to equation (66) by choosing the pilot vector $\bar{s}$ in the x-direction, normal to the interface. The solutions to Maxwell's equations in this case are

$$
\begin{bmatrix} \Phi \\ A_z \\ A_y \\ A_x \end{bmatrix} =
\begin{bmatrix} i\omega\psi_i \\ \partial\psi_i/\partial z \\ \partial\psi_i/\partial y \\ \partial\psi_i/\partial x \end{bmatrix} ,
\begin{bmatrix} 0 \\ -\partial\psi_i/\partial y \\ \partial\psi_i/\partial z \\ 0 \end{bmatrix} ,
\begin{bmatrix} 0 \\ \partial^2\psi_i/\partial x\partial z \\ \partial^2\psi_i/\partial x\partial y \\ -\left(\partial^2\psi_i/\partial y^2 + \partial^2\psi_i/\partial z^2\right) \end{bmatrix} ,
\tag{67}
$$

where y and z are directions that are locally tangent to the interface. Substitution of the second and third solutions of equation (67) into the equations for the electric and magnetic fields (eq (29)) leads to sets of solutions with electric or magnetic field components of zero in the x-direction. This property was used by Debye to define these potentials [5]. By using the continuity of tangential components of E and H at a dielectric interface, Debye derived boundary conditions that the potentials obey at the surface of a sphere of radius a as

$$
\begin{aligned}
\frac{\partial^2 r\psi_{11}(k_1 a)}{\partial r^2} &= \frac{\partial^2 r\psi_{21}(k_2 a)}{\partial r^2} \\
\varepsilon_1\, \psi_{11}(k_1 a) &= \varepsilon_2\, \psi_{21}(k_2 a) \\
\frac{\partial^2 r\psi_{12}(k_1 a)}{\partial r^2} &= \frac{\partial^2 r\psi_{22}(k_2 a)}{\partial r^2} \\
\mu_1\, \psi_{12}(k_1 a) &= \mu_2\, \psi_{22}(k_2 a) \ ,
\end{aligned}
\tag{68}
$$

where the first subscript refers to a dielectric region and the second sub-script refers to the scalar solution to the wave equation. Debye computed scattering intensities by constructing solutions that obeyed the radiation condition in the outer region that were finite in the inner region, and that obeyed the boundary conditions of equation (68). If the vector $\bar{s}$ lies in the z-direction in the plane of the interface, a set of solutions is

$$
\begin{bmatrix} \Phi \\ A_z \\ A_y \\ A_x \end{bmatrix} = \begin{bmatrix} i\omega\psi_i \\ -i\omega_z\psi_i \\ \partial\psi_i/\partial y \\ \partial\psi_i/\partial x \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\partial\psi_i/\partial x \\ \partial\psi_i/\partial y \end{bmatrix}, \begin{bmatrix} 0 \\ i\left(k_i/\omega_z\right)\psi_i \\ \partial\psi_i/\partial y \\ \partial\psi_i/\partial x \end{bmatrix}.
\tag{69}
$$

If one of the scalar potentials is taken to be $u\ (x,\ y)$ and the other $v\ (x,\ y)$, these solutions can be combined to give the potential forms of equations (17) and (22). For equation (17), the gauge solution is combined with the last solution (both with scalar function $u\ (x,\ y)$, and the combination is added to the second solution, with scalar function $v\ (x,\ y)$. The forms of equation (22) are found in a similar way.

The coupled azimuthal potentials developed by Morgan, et al [10] can also be described by equation (66). They were developed using cylindrical coordinates, although the development can be extended to other coordinate systems. In the notation and unit system used here, fields resulting from the coupled azimuthal potentials are

$$
\frac{1}{rf_m} E_r = \frac{1}{r}\left(\frac{\partial^2\psi_1}{\partial r\partial\phi} - i\omega\mu\frac{\partial\psi_2}{\partial z}\right) \qquad \frac{1}{rf_m} H_r = \frac{1}{r}\left(\frac{\partial^2\psi_2}{\partial r\partial\phi} + i\omega\varepsilon\frac{\partial\psi_1}{\partial z}\right)
$$

$$
\frac{1}{rf_m} E_z = \frac{1}{r}\left(\frac{\partial^2\psi_1}{\partial z\partial\phi} + i\omega\mu\frac{\partial\psi_2}{\partial r}\right) \qquad \frac{1}{rf_m} H_z = \frac{1}{r}\left(\frac{\partial^2\psi_2}{\partial z\partial\phi} - i\omega\varepsilon\frac{\partial\psi_1}{\partial r}\right)
\tag{70}
$$

$$
\frac{1}{rf_m} E_\phi = \frac{1}{c^2}\psi_1 + \frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\phi^2} \qquad \frac{1}{rf_m} H_\phi = \frac{1}{c^2}\psi_2 + \frac{1}{r^2}\frac{\partial^2\psi_2}{\partial\phi^2},
$$

where

$$
f_m = \left[(\omega r/c)^2 - m^2\right]^{-1},
\tag{71}
$$

and the azimuthal dependence is of the form exp(imϕ), where m is an integer. The permittivity ε and the permeability μ are assumed to be functions of the coordinates $(r,\ z)$. If they were constant, and if the quantity $(rf_m)^{-1}$ were not in the definition, equation (70), without this factor, could be derived from potentials of the form of equation (66), with the pilot vector $\bar{s}$ in the ϕ direction. In this case, the potentials would not be coupled, but would satisfy the wave equation individually.

Bouwkamp and Casimer [18] derived a relationship between the Debye potentials and current distributions. They showed that, if the fields of the eigenfunctions of a dielectric sphere were generated by currents, the fields resulting from an electric multipole distribution of currents would be derivable from one of the Debye potentials, and those fields resulting from

magnetic multipole distributions would be derivable from the other. The scalar potentials $u(x, y)$ and $v(x, y)$ can be related to surface currents. Components of equation (21) satisfy the boundary conditions of equation (11) and can be associated with currents. These components can be substituted into equation (61). The first component is the gauge solution and leads to charges and currents of zero. The second component leads to the equations for surface currents

$$-\frac{\omega}{c^2}\frac{\partial u}{\partial x} = \mu j_z$$
$$0 = \mu j_y \, ,$$

(72)

and the third leads to

$$-\frac{\omega_z^2 \, \omega}{(ck)^2}\frac{\partial v}{\partial y} = \mu j_z$$
$$i\frac{\omega}{c^2}\omega_z v = \mu j_y \, .$$

(73)

8. Conclusions

The preceding sections describe an approach for computing electric and magnetic fields using scalar solutions to the wave equation that are zero at an interface and have normal derivatives of zero at the interface. The use of these two functions simplifies their matching at the interface. The scalar functions are computed from the locations of the interface and the constants (ε, μ) that characterize the regions. Boundary conditions for the electric and magnetic fields have been satisfied by forming linear combinations of pairs of scalar functions that solve the wave equation.

The computations presented here are two-dimensional. At a dielectric interface, it is assumed that the ratio q of the derivatives of the scalar functions is constant along the boundary of the two-dimensional region, and that the normal derivatives of u are proportional on each side of the interface. These conditions imply that values of v are also proportional on each side of the interface. The coordinate system must be chosen to ensure that these conditions hold. These conditions replace the matching formulas of equation (68).

Many analyses of boundary conditions at a conducting surface have been carried out in terms of $u\,(x, y)$ and $v\,(x, y)$, although these functions are not regarded as scalar potentials, but as representations of tangential electric and magnetic fields, since, in this case, the tangential fields obey exactly the same boundary conditions. It has been shown here that these scalar potentials can also be applied to dielectric interfaces, and that they can assist in simplifying computations in this case.

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